

# UNIFORM ALGEBRAS AND APPROXIMATION ON MANIFOLDS

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**ABSTRACT.** Let  $\Omega \subset \mathbb{C}^n$  be a bounded domain and let  $\mathcal{A} \subset \mathcal{C}(\overline{\Omega})$  be a uniform algebra generated by a set  $F$  of holomorphic and pluriharmonic functions. Under natural assumptions on  $\Omega$  and  $F$  we show that the only obstruction to  $\mathcal{A} = \mathcal{C}(\overline{\Omega})$  is that there is a holomorphic disk  $D \subset \overline{\Omega}$  such that all functions in  $F$  are holomorphic on  $D$ , i.e., the only obstruction is the obvious one. This generalizes work by A. Izzo. We also have a generalization of Wermer's maximality theorem to the (distinguished boundary of the) bidisk.

## 1. INTRODUCTION

In this article we will discuss versions of two theorems of John Wermer: His well known maximality theorem [14] states that if  $f \in \mathcal{C}(b\mathbb{D})$ , then either  $f$  is the boundary value of a holomorphic function on the disk, or  $[z, f]_{b\mathbb{D}} = \mathcal{C}(b\mathbb{D})$ . Here  $[z_1, f]_{b\mathbb{D}}$  denotes the uniform algebra generated by  $z$  and  $f$  on the boundary of the disk. A closely related result is the following [15]: Let  $f \in \mathcal{C}^1(\overline{\mathbb{D}})$  and assume that the graph  $\mathcal{G}_f(\overline{\mathbb{D}})$  of  $f$  over  $\overline{\mathbb{D}}$  in  $\mathbb{C}^2$  is polynomially convex. Let  $S := \{z \in \overline{\mathbb{D}} : \partial f(z) = 0\}$ . Then

$$[z_1, f]_{\overline{\mathbb{D}}} = \{g \in \mathcal{C}(\overline{\mathbb{D}}) : g|_S \in \mathcal{O}(S)\},$$

Note that if  $f$  is harmonic, then  $f$  is holomorphic or  $\mathcal{O}(S) = \mathcal{C}(S)$ .

We let  $PH(\Omega)$  denote the pluriharmonic functions on a domain  $\Omega \subset \mathbb{C}^n$ , and we let  $\Gamma^2$  denote the distinguished boundary of the bidisk  $\mathbb{D}^2$ .

Our most complete results are in  $\mathbb{C}^2$  and are contained in the following two theorems:

**Theorem 1.1.** *Let  $h_j \in PH(\mathbb{D}^2) \cap \mathcal{C}^1(\overline{\mathbb{D}^2})$  for  $j = 1, \dots, N$ . Then either there exists a holomorphic disk in  $\overline{\mathbb{D}^2}$  where all  $h_j$ -s are holomorphic, or  $[z_1, z_2, h_1, \dots, h_N]_{\overline{\mathbb{D}^2}} = \mathcal{C}(\overline{\mathbb{D}^2})$ .*

**Theorem 1.2.** *Let  $f_j \in \mathcal{C}(\Gamma^2)$ ,  $j = 1, \dots, N$ ,  $N \geq 1$ , and assume that each  $f_j$  extends to a pluriharmonic function on  $\mathbb{D}^2$ . Then either*

$$[z_1, z_2, f_1, \dots, f_N]_{\Gamma^2} = \mathcal{C}(\Gamma^2),$$

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or there exists a nontrivial closed algebraic variety  $Z \subset \overline{\mathbb{D}^2} \setminus \Gamma^2$  with  $\overline{Z} \setminus Z \subset \Gamma^2$ , and the pluriharmonic extensions of the  $f_j$ -s are holomorphic on  $Z$ .

(We will give an example (Example 5.7) with  $N = 1$ , where no such variety can exist, i.e.,  $\mathcal{C}(\Gamma^2) = [z_1, z_2, f]$ .)

**Theorem 1.3.** *Let  $\Omega \subset \mathbb{C}^n$  be a polynomially convex  $\mathcal{C}^1$ -smooth domain. Let  $h_j \in PH(\Omega) \cap \mathcal{C}(\overline{\Omega})$  for  $j = 1, \dots, N$ , and assume that  $[z_1, \dots, z_n, h_1, \dots, h_N]_{b\Omega} = \mathcal{C}(b\Omega)$ . Then either there exists a holomorphic disk in  $\Omega$  where all  $h_j$ -s are holomorphic, or  $[z_1, \dots, z_n, h_1, \dots, h_N]_{\overline{\Omega}} = \mathcal{C}(\overline{\Omega})$ .*

We remark that if  $\Omega$  was strictly pseudoconvex, we would get the same result with the algebra  $\mathcal{A}(\overline{\Omega})[h_1, \dots, h_N]$  instead.

In the one-variable case, Theorem 1.1 is due to Čirca [3] (see also Axler-Shields [1]). Theorems 1.1 and 1.3 generalize work by A. Izzo [7] and Weinstock [13]. Theorem 1.2 generalizes Wermer's maximality theorem to  $\mathbb{C}^2$  to the effect that analyticity is the only obstruction to the full algebra being generated.

Assume in addition to the conditions in Theorem 1.3 that  $\Omega$  is strictly pseudoconvex and that  $h_j \in \mathcal{C}^1(\overline{\Omega})$  for  $j = 1, \dots, N$ , and define

$$S := \{z \in \overline{\Omega} : \bar{\partial}h_{i_1} \wedge \dots \wedge \bar{\partial}h_{i_n}(z) = 0 \text{ for all } 1 \leq i_j \leq N\}.$$

Izzo's result is that if  $S \cap \Omega = \emptyset$  and  $N = n$  then  $[z_1, \dots, z_n, h_1, \dots, h_n]_{\overline{\Omega}} = \mathcal{C}(\overline{\Omega})$ . In this case  $S \subset b\Omega$  is a peak interpolation set by Weinstock [12], and by the assumption on the wedge products we have that  $\mathcal{G}_h(\overline{\Omega} \setminus S)$  is totally real. The pluriharmonicity of the  $h_j$ -s guaranties that  $\mathcal{G}_h(\overline{\Omega})$  is polynomially convex, hence the conclusion follows from Theorem 3.5 below.

According to Theorem 1.3 there is no need in general to assume that  $S \cap \Omega = \emptyset$ . For a generic choice of  $h_j$ -s (as long as  $N \geq n$ ) we have that  $S \cap \Omega \neq \emptyset$  will not prevent the full algebra from being generated, but in exceptional cases the presence of a nontrivial analytic set  $Z \subset \Omega$  with all  $h_j$ -s analytic along  $Z$  will be an obstruction. We have computational criteria for detecting such a set. For approximation of continuous functions on  $\mathcal{G}_h(\overline{\Omega})$  it is then necessary and sufficient to assume that the function sought approximating is approximable on  $\mathcal{G}_h(Z \cup b\Omega)$  (as above, the boundary might be covered by other existing results).

## 2. PROOF OF A THEOREM OF A. IZZO

Recently A. Izzo [8] proved a conjecture of Freeman [4] regarding uniform algebras on manifolds. At the core of our proofs of the theorems in our introduction is an approximation result due to P. E. Manne [9] concerning  $\mathcal{C}^1$ -approximation by holomorphic functions on totally real sets attached to compact sets (cf. [10]). We will demonstrate its strength by giving a very short proof of the result of Izzo. Whereas Izzo uses the Arens-Calderón

lemma to utilize techniques of Weinstock [13] (cf. Berndtsson [2]), we will use it to utilize techniques of Manne [9] (cf. Manne, Wold and Øvrelid [10]).

**Theorem 2.1.** (*A. Izzo*) *Let  $M$  be an  $m$ -dimensional  $\mathcal{C}^1$ -manifold-with-boundary, and let  $X$  be a compact subset of  $M$ . Let  $A$  be a uniform algebra on  $X$  generated by a family  $\Phi$  of complex valued functions  $\mathcal{C}^1$  on  $M$ , assume that the maximal ideal space of  $A$  is  $X$ , and let*

$$E := \{p \in X : df_1 \wedge \cdots \wedge df_m(p) = 0 \text{ for all } f_1, \dots, f_m \in \Phi\}.$$

*Then  $A = \{g \in \mathcal{C}(X) : g|_E \in A|_E\}$ .*

*Proof.* It is enough to show that for any point  $x \notin E$  there exists an open neighborhood  $D$  of  $x$  such that any continuous function on  $X$  with compact support in  $D$  is in  $A$ . By definition (see Lemma 3.1) there exist  $f_1, \dots, f_n \in A$  and a neighborhood  $D'$  of  $x$  such that  $F(D' \cap X)$  is a totally real set, where  $F = (f_1, \dots, f_n)$ . Since  $A$  needs to separate points of  $X$  (since the maximal ideal space of  $A$  is  $X$ ) we may, by possibly having to add more functions, assume that  $F^{-1}(F(X) \cap F(D' \cap X)) = D' \cap X$ . Let  $\Omega$  be a neighborhood of  $X_0 = F(X)$  as in Lemma 2.2. Since the maximal ideal space of  $A$  is  $X$  it follows from the Arens-Calderón lemma (see *e.g.* [6]) that there exist functions  $f_{n+1}, \dots, f_{n+m} \in A$  such that, writing  $\tilde{F} = (f_1, \dots, f_n, f_{n+1}, \dots, f_{n+m})$ , we have that  $\pi_n(\widehat{\tilde{F}(X)}) \subset \Omega$ . The result now follows from Lemma 2.2.  $\square$

**Lemma 2.2.** *Let  $X_0 \subset \mathbb{C}^n$  be a compact set and assume that  $z_0 \in X_0$  is a totally real point, i.e.,  $X_0$  is a totally real set near  $z_0$ . Then there exists a neighborhood  $\Omega$  of  $X_0$  and a neighborhood  $U'$  of  $z_0$  such that the following holds: Let  $\pi_n : \mathbb{C}^n \times \mathbb{C}^m \rightarrow \mathbb{C}^n$  denote the projection onto the first  $n$  coordinates, let  $X_1 \subset \mathbb{C}^{n+m}$  be compact with  $\pi_n(X_1) = X_0$ , and assume that  $\pi_n(\widehat{X_1}) \subset \Omega$ . Then for any  $f \in \mathcal{C}_0(U' \cap X_0)$  we have that*

$$(f \circ \pi_n)|_{X_1} \in [z_1, \dots, z_{n+m}]_{X_1}.$$

*Proof.* We follow the proof of Proposition 3.10 in [10]. Let  $V$  be a neighborhood of  $z_0$  such that  $V \cap X_0$  is totally real. The following is the content of the proposition on page 522 in [9]: There are neighborhoods

$$U' \subset \subset U'' \subset \subset U \subset \subset V$$

of  $z_0$ , and a neighborhood  $W \subset U$  of  $X_0 \cap bU''$  such that if  $f \in \mathcal{C}(X_0 \cap V)$  has compact support in  $U'$  there exists a sequence of holomorphic functions  $h_j \in \mathcal{O}(U)$  such that  $\|h_j - f\|_{X_0 \cap U} \rightarrow 0$  and  $\|h_j\|_W \rightarrow 0$  uniformly as  $j \rightarrow \infty$ . Let  $\{\Omega_j\}$  be a neighborhood basis of  $X_0$  and define  $U_j^1 = \Omega_j \cap U''$ ,  $U_j^2 = (\Omega_j \setminus U'') \cup (W \cap \Omega_j)$ . If  $j$  is large enough, we have that  $U_j^2$  is open, and  $\Omega_j = U_j^1 \cup U_j^2$  and  $U_j^1 \cap U_j^2 \subset W$ . Fix a  $j$  large enough so that this holds and drop the subscript  $j$ .

Assume that  $\pi_n(\widehat{X_1}) \subset \Omega$ , and choose a Runge and Stein neighborhood  $\tilde{\Omega}$  of  $X_1$  with  $\pi_n(\tilde{\Omega}) \subset \Omega$ . We solve a Cousin problem on  $\tilde{\Omega}$  with respect to the cover  $\tilde{U}^1 = \tilde{\Omega} \cap \pi_n^{-1}(U^1)$  and  $\tilde{U}^2 = \tilde{\Omega} \cap \pi_n^{-1}(U^2)$ .

Let  $\tilde{h}_j = h_j \circ \pi_n$ . By the solution of Cousin I with estimates there exist sequences  $g_j^i \in \mathcal{O}(\tilde{U}^i)$  such that  $\tilde{h}_j = g_j^1 - g_j^2$  on  $\tilde{U}^1 \cap \tilde{U}^2$ , and such that  $g_j^i \rightarrow 0$  uniformly on compact subsets of  $\tilde{U}^i$ . The sequence  $g_j$  defined as  $h_j - g_j^1$  on  $\tilde{U}^1$  and  $-g_j^2$  on  $\tilde{U}^2$  will converge to  $f \circ \pi_n|_{X_1}$ .  $\square$

**Remark 2.3.** We remark that Izzo's somewhat more general Theorem 1.2. can be proved in a similar fashion.

### 3. PRELIMINARIES

#### 3.1. Toally real manifolds.

**Lemma 3.1.** *Let  $M$  be a smooth manifold of real dimension  $m$  and let  $f = (f_1, \dots, f_N): M \rightarrow \mathbb{C}^N$  be a  $\mathcal{C}^1$ -smooth map. If  $x \in M$  and if there are  $f_{i_1}, \dots, f_{i_m}$  such that*

$$df_{i_1} \wedge \dots \wedge df_{i_m}(x) \neq 0,$$

*then the map  $f$  is an embedding near  $x$  and the image  $T_{f(x)}f(M)$  is totally real.*

*Proof.* The first assertion is clear (actually the condition that the wedge product is non-vanishing is stronger than the condition that  $f$  is an embedding near  $x$ ). This means that the  $\mathbb{C}$ -linear map  $df: T_x^{\mathbb{C}}M \rightarrow \mathbb{C}^N$  has maximal rank, where  $T_x^{\mathbb{C}}M$  is the complexified tangent space of  $M$  at  $x$ . Hence,  $T_{f(x)}f(M) \cap JT_{f(x)}f(M) = df(T_xM \cap JT_xM) = 0$ .  $\square$

**Corollary 3.2.** *Let  $X$  be a complex manifold of dimension  $n$ , let  $h = (h_1, \dots, h_N): X \rightarrow \mathbb{C}^N$  be a  $\mathcal{C}^1$ -smooth map, and let  $\mathcal{G}_h(X) \subset X \times \mathbb{C}^N$  be the graph of  $h$ . If  $x \in X$  and there are  $h_{i_1}, \dots, h_{i_n}$  such that*

$$\bar{\partial}h_{i_1} \wedge \dots \wedge \bar{\partial}h_{i_n}(x) \neq 0,$$

*then the tangent space of  $\mathcal{G}_h(X)$  at  $(x, h(x))$  is totally real.*

*Proof.* Consider the map  $X \rightarrow X \times \mathbb{C}^N$  defined by  $z \mapsto (z, h(z))$  and denote by  $dz$  the form  $dz_1 \wedge \dots \wedge dz_n$  where  $z_j$  are local coordinates centered at  $x$ . For bidegree reasons

$$dz \wedge dh_{i_1} \wedge \dots \wedge dh_{i_n} = dz \wedge \bar{\partial}h_{i_1} \wedge \dots \wedge \bar{\partial}h_{i_n},$$

and by assumption the latter product is non-vanishing at  $x$ .  $\square$

**3.2. Pluriharmonic functions.** Let  $h = (h_1, \dots, h_N): \Omega \rightarrow \mathbb{C}^N$ ,  $n \leq N$ , be a pluriharmonic mapping. Let  $e_1, \dots, e_N$  be a basis for  $\mathbb{C}^N$ . Consider  $\{e_j\}$  as a frame for the trivial rank  $N$ -bundle  $E \rightarrow \Omega$  and consider  $h = h_1e_1 + \dots + h_Ne_N$  as a section of  $E$ . We let  $H_1$  be the section of  $T_{0,1}^*(\Omega) \wedge E \simeq T_{0,1}^*(\Omega) \otimes E$  defined by

$$(3.1) \quad H_1 = \bar{\partial}h = \bar{\partial}h_1 \wedge e_1 + \dots + \bar{\partial}h_N \wedge e_N$$

and we define  $H_k$  as sections of  $T_{0,k}^*(\Omega) \wedge \Lambda^k E$  by

$$(3.2) \quad H_k = (H_1)^k / k! = H_1 \wedge \cdots \wedge H_1 / k! = \pm \sum_{|I|=k}^I \bar{\partial} h_I \wedge e_I,$$

where  $\sum_I$  means that we sum over increasing multiindices and  $\bar{\partial} h_I = \bar{\partial} h_{I_1} \wedge \cdots \wedge \bar{\partial} h_{I_k}$ . Since the  $H_k$  are invariantly defined, this construction also makes sense for pluriharmonic mappings from complex manifolds  $X \rightarrow \mathbb{C}^N$ . Moreover,  $H_k$  is anti-holomorphic, or more formally, an anti-holomorphic  $(0, k)$ -form with values in  $\Lambda^k E$ . If  $i: Y \hookrightarrow X$  is a  $k$ -dimensional complex submanifold, we will write  $Y_H^k$  for the set of points of  $Y$  where  $i^* H_k$  vanishes; if  $\zeta_1, \dots, \zeta_k$  are local coordinates on  $Y$  this set coincides with the common zero set of the tuple of all  $k \times k$ -subdeterminants of the matrix

$$\begin{pmatrix} \partial h_1 / \partial \bar{\zeta}_1 & \cdots & \partial h_1 / \partial \bar{\zeta}_k \\ \vdots & & \vdots \\ \partial h_N / \partial \bar{\zeta}_1 & \cdots & \partial h_N / \partial \bar{\zeta}_k \end{pmatrix}.$$

Hence,  $Y_H^k$  is an analytic subset of  $Y$ .

**Proposition 3.3.** *Let  $X$  be a complex manifold of dimension  $n$  and let  $h = (h_1, \dots, h_N): X \rightarrow \mathbb{C}^N$ ,  $N \geq n$ , be a pluriharmonic mapping. If there is a (germ of a) complex submanifold  $Y \subset X$  of dimension  $k$  such that  $Y_H^k = Y$ , then there is a (germ of a) holomorphic disk in  $Y$  where all  $h_j$  are holomorphic.*

*Proof.* It suffices to show that if  $H_n$  vanishes identically, then  $X$  contains a holomorphic disk where all  $h_j$  are holomorphic. We prove this by induction over  $n$ . If  $n = 1$ , then the condition that  $H_1$  vanishes identically precisely means that all  $h_j$  are holomorphic. Assume that the statement is true for all  $n \leq k$  and all  $N \geq n$ . Let  $X$  be  $k+1$ -dimensional and assume that  $h: X \rightarrow \mathbb{C}^N$ ,  $N \geq k+1$ , is pluriharmonic and that  $H_{k+1}$  vanishes identically. We may assume that there is some  $h_j$  which is not holomorphic on  $X$ . Assume for simplicity that  $h_1$  is not holomorphic and let  $x \in X$  be a point such that  $\bar{\partial} h_1(x) \neq 0$ . In a neighborhood of  $x$  we may write

$$h_1 = \Re g + f,$$

where  $g$  and  $f$  are holomorphic. Since  $\bar{\partial} h_1(x) \neq 0$  it follows that  $dg(x) \neq 0$ . Let  $Y = \{z; g(z) = g(x)\}$  be the level set of  $g$  through  $x$ . Then  $i: Y \hookrightarrow X$  is a complex  $k$ -dimensional submanifold through  $x$ . Choose local coordinates  $(w_0, w_1, \dots, w_k) = (w_0, w')$  centered at  $x$  such that  $Y = \{w_0 = 0\}$ . Since  $g$  is constant on  $Y$  we have  $i^* \bar{\partial} h_1 = i^* \bar{\partial} \Re g = 0$ , i.e.,  $h_1$  is holomorphic on  $Y$ . Thus, since  $\bar{\partial} h_1(x) \neq 0$  it follows that

$$\frac{\partial h_1}{\partial \bar{w}_0}(x) \neq 0, \quad \frac{\partial h_1}{\partial \bar{w}_j} = 0, \quad j = 1, \dots, k, \quad \text{on } Y.$$

Now, let  $I \subset \{1, \dots, N\}$ ,  $|I| = k$ . If  $1 \in I$ , then it is obvious that  $i^* \bar{\partial} h_I = i^* \bar{\partial} h_{I_1} \wedge \dots \wedge \bar{\partial} h_{I_k} = 0$  so we may assume that  $1 \notin I$ . Since  $H_{k+1}$  vanishes identically we have

$$0 = \det \begin{pmatrix} \partial h_1 / \partial \bar{w}_0 & 0 \\ * & \partial h_I / \partial \bar{w}' \end{pmatrix} = \frac{\partial h_1}{\partial \bar{w}_0} \cdot \det \left( \frac{\partial h_I}{\partial \bar{w}'} \right)$$

on  $Y$  and since  $\partial h_1 / \partial \bar{w}_0 \neq 0$  close to  $x$  it follows that  $i^* \bar{\partial} h_I = 0$ . Hence,  $i^* H_k$  vanishes identically in a neighborhood in  $Y$  of  $x$  and it follows from the induction hypothesis that  $Y$  contains a holomorphic disk where all  $h_j$  are holomorphic.  $\square$

**3.3. Polynomial convexity and approximation on stratified totally real sets.** We now consider approximation on stratified totally real sets. The following result gives a sufficient condition for when a compact polynomially convex set  $X \subset \mathbb{C}^n$  has the property that  $\mathcal{C}(X) = [z_1, \dots, z_n]_X$ . The technical and main part of the proof is contained in [10, Proposition 3.13]. For convenience of the reader we state here a simplified version of this proposition:

**Proposition 3.4.** [10, Proposition 3.13] *Let  $K \subset \mathbb{C}^n$  be a compact set,  $M \subset \mathbb{C}^n$  a totally real set,  $M_0 \subset M$  compact, and assume that  $K \cup M_0$  is polynomially convex. Then for any  $f \in \mathcal{C}(K \cup M_0)$  with  $\text{Supp}(f) \cap K = \emptyset$ , there exists a sequence  $\{h_j\}_{j=1}^\infty \subset \mathcal{O}(\mathbb{C}^n)$  such that  $\|h_j - f\|_{K \cup M_0} \rightarrow 0$  as  $j \rightarrow \infty$ .*

**Theorem 3.5.** *Let  $X$  be a polynomially convex compact set in  $\mathbb{C}^n$  and assume that there are closed sets  $X_0 \subset \dots \subset X_N = X$  such that*

- (i)  $X_j \setminus X_{j-1}$ ,  $j = 1, \dots, N$ , *is a totally real set.*

*Then for  $f \in \mathcal{C}(X) \cap \mathcal{O}(X_0)$  we have that  $f \in [z_1, \dots, z_n]_X$ . In particular, if  $\mathcal{C}(X_0) = \mathcal{O}(X_0)$  then  $\mathcal{C}(X) = [z_1, \dots, z_n]_X$ .*

*Proof.* We notice that each  $X_j$  has to be polynomially convex. In fact, we trivially have  $\widehat{X_{N-1}} \subseteq \widehat{X_N} = X_N$ . Moreover, if  $x \in X_N \setminus X_{N-1}$  then, by Proposition 3.4, there is a polynomial  $P_x$  such that  $\|P_x\|_{X_{N-1}} < |P_x(x)|$ . Hence,  $x \notin \widehat{X_{N-1}}$  and so  $\widehat{X_{N-1}} = X_{N-1}$ . Repeating the argument we see that  $X_{N-2}$  is polynomially convex, and so on.

Now let  $f \in \mathcal{C}(X) \cap \mathcal{O}(X_0)$ . Proceeding by induction we will show that if  $f \in \mathcal{O}(X_k)$  then  $f \in \mathcal{O}(X_{k+1})$  for  $k \geq 0$ . Let  $\epsilon > 0$ , and let  $g \in \mathcal{O}(\mathbb{C}^n)$  with  $\|g - f\|_{X_k} < \frac{\epsilon}{4}$ . Let  $U$  be an open neighborhood of  $X_k$  such that  $\|g - f\|_{\bar{U} \cap X_{k+1}} < \frac{\epsilon}{2}$ . Let  $\chi \in C_0^\infty(U)$  with  $0 \leq \chi \leq 1$  and  $\chi \equiv 1$  near  $X_k$ . Then

$$g + (1 - \chi) \cdot (f - g)$$

is an  $\epsilon$ -approximation of  $f$  on  $X_{k+1}$ ,  $g$  is entire, and  $(1 - \chi) \cdot (f - g)$  is approximable on  $X_{k+1}$  by entire functions by Proposition 3.4.  $\square$

**Lemma 3.6.** *Let  $\Omega \subset \mathbb{C}^n$  be a  $\mathcal{C}^1$ -smooth polynomially convex domain, and let  $h_j \in PH(\Omega) \cap \mathcal{C}(\overline{\Omega})$  for  $j = 1, \dots, N$ . Then  $\mathcal{G}_h(\overline{\Omega})$  is polynomially convex.*

*Proof.* Let  $(z_0, w_0) \in (\mathbb{C}^n \times \mathbb{C}^N) \setminus \mathcal{G}_h(\overline{\Omega})$ . If  $z_0 \notin \overline{\Omega}$  it is clear that  $(z_0, w_0) \notin \widehat{\mathcal{G}_h(\overline{\Omega})}$  since  $\overline{\Omega}$  is polynomially convex. Assume that  $z_0 \in \overline{\Omega}$ . For some  $j$  we have  $\Re(w_{0,j} - h_j(z_0)) \neq 0$  and we may assume that  $g(z, w) := \Re(w_j - h_j(z))$  is positive at  $(z_0, w_0)$ . Then  $g$  is pluriharmonic in  $\Omega$ , continuous up to the boundary, and satisfies  $g(z_0, w_0) > \|g\|_{\mathcal{G}_h(\overline{\Omega})} = 0$ . By [5, Theorem 1], the function  $g$  is uniformly approximable on  $\overline{\Omega}$  by functions in  $PSH(\Omega) \cap \mathcal{C}^\infty(\overline{\Omega})$ . Thus, since in addition  $\overline{\Omega}$  is polynomially convex, it follows that there is an open Runge and Stein neighborhood  $\tilde{\Omega} \supset \overline{\Omega}$  and a function  $\tilde{g} \in PSH(\tilde{\Omega} \times \mathbb{C}^N)$  such that  $\tilde{g}(z_0, w_0) > \|\tilde{g}\|_{\mathcal{G}_h(\overline{\Omega})}$ . It thus follows that  $(z_0, w_0) \notin \widehat{\mathcal{G}_h(\overline{\Omega})}_{\mathcal{O}(\tilde{\Omega} \times \mathbb{C}^N)}$  and so  $(z_0, w_0) \notin \widehat{\mathcal{G}_h(\overline{\Omega})}$ .  $\square$

The first part of the following result can be found in [10].

**Proposition 3.7.** *Let  $K \subset \mathbb{C}^N$  be compact, let  $F : \mathbb{C}^N \rightarrow \mathbb{C}^M$  be the uniform limit on  $K$  of entire functions, and let  $Y = F(K)$ ; note that  $F$  extends to  $\widehat{K}$ . For a point  $y \in Y$ , let  $F_y$  denote the fiber  $F^{-1}(y) \subset \widehat{K}$ , and let  $K_y$  denote the restricted fiber  $F_y \cap K$ . The following holds:*

- i) *if  $y$  is a peak point for the algebra  $[z_1, \dots, z_M]_Y$ , then  $\widehat{K} \cap F_y = \widehat{K}_y$ , and*
- ii) *if  $[z_1, \dots, z_M]_Y = \mathcal{C}(Y)$  then*

$$[z_1, \dots, z_N]_K = \{f \in \mathcal{C}(K) : f|_{K_y} \in [z_1, \dots, z_N]_{K_y} \text{ for all } y \in Y\}.$$

**Remark 3.8.** Note that ii) is a consequence of Bishop's antisymmetric decomposition theorem. The proof we will give here is due to Nils Øvrelid, and is almost trivial.

*Proof.* For the proof of i) see Proposition 4.3 in [10]. To prove ii) simply glue together functions which are good near the fibers  $K_y$  using a continuous partition of unity on the projection  $Y = F(K)$ .  $\square$

#### 4. PROOF OF THEOREM 1.1

We first note that the graph  $\mathcal{G}_h(\overline{\mathbb{D}^2})$  is polynomially convex. The proof of Lemma 3.6 goes through except for that we, strictly speaking, cannot use Theorem 1 in [5] to conclude that the function  $g$  is uniformly approximable on  $\overline{\mathbb{D}^2}$  by functions in  $PSH(\mathbb{D}^2) \cap \mathcal{C}^\infty(\overline{\mathbb{D}^2})$  since  $b\mathbb{D}^2$  is not smooth. However, this is obvious since  $\mathbb{D}^2$  is starshaped;  $g$  can even be approximated uniformly on  $\overline{\mathbb{D}^2}$  by functions pluriharmonic in a neighborhood of  $\overline{\mathbb{D}^2}$ .

Assume there is no holomorphic disk in  $\overline{\mathbb{D}^2}$  where all  $h_j = h_j(z, w)$  are holomorphic. We will show that the polynomials in  $\mathbb{C}^2 \times \mathbb{C}^N$  are dense in  $\mathcal{C}(\mathcal{G}_h(\overline{\mathbb{D}^2}))$ .

The part of the boundary  $b\mathbb{D}^2 \setminus \Gamma^2$  is the disjoint union  $(\mathbb{D} \times S^1) \cup (S^1 \times \mathbb{D})$ . Consider the part  $\mathbb{D} \times S^1 \subset b\mathbb{D}^2 \setminus \Gamma^2$ ; the other part is treated in a completely analogous way. Let  $g_j$  be the complex conjugate of the restriction of  $\partial h_j / \partial \bar{z}$  to  $\mathbb{D} \times S^1$ . Then  $g_j(z, s) \in \mathcal{C}(\overline{\mathbb{D}} \times S^1)$  is holomorphic in  $\mathbb{D}$  for each fixed  $s \in S^1$ . Moreover, by expressing  $h_j(\cdot, s)$  as a Poisson integral and differentiating under the integral sign, it follows that  $g_j \in \mathcal{C}^1(\mathbb{D} \times S^1)$ . Let

$$Z = \{(z, s) \in \mathbb{D} \times S^1; g_j(z, s) = 0, \forall j\}.$$

Then, by Corollary 3.2, the graph of  $h$  over  $(\mathbb{D} \times S^1) \setminus Z$  is a totally real manifold.

**Lemma 4.1.** *There are closed sets  $B \subset E \subset \mathbb{D} \times S^1$  such that*

- (i)  $Z \subset E$
- (ii)  $E \setminus B$  is a  $\mathcal{C}^1$ -smooth manifold of real dimension 1
- (iii) for any  $\delta > 0$ , one can cover  $B$  by the union of finitely many pairwise disjoint open sets  $U_0, \dots, U_k$  such that  $\text{diam}(U_j) < \delta$ ,  $j = 1, \dots, k$  and  $U_0 \subset \{(z, s) \in \mathbb{D} \times S^1; |z| > 1 - \delta\}$ .

We take this lemma for granted for the moment and finish the proof of Theorem 1.1. We define a stratification  $Y_{-3} \subset \dots \subset Y_2 = \overline{\mathbb{D}^2}$ ; cf., Section 6. Let

$$\begin{aligned} Y_2 &= \overline{\mathbb{D}^2}, \\ Y_1 &= b\mathbb{D}^2 \cup \{z \in \mathbb{D}^2; H_2(z) = 0\} =: b\mathbb{D}^2 \cup Z_1, \\ Y_0 &= b\mathbb{D}^2 \cup \text{Sing}(Z_1) \cup \text{Reg}(Z_1)_H^1, \\ Y_{-1} &= b\mathbb{D}^2, \\ Y_{-2} &= \Gamma^2 \cup E \cup E', \\ Y_{-3} &= \Gamma^2 \cup B \cup B', \end{aligned}$$

where  $B' \subset E' \subset S^1 \times \mathbb{D} \subset b\mathbb{D}^2$  are analogous to  $B \subset E$ ; see Subsection 3.2 for the definition of  $\text{Reg}(Z_1)_H^1$ . Since there is no holomorphic disk in  $\overline{\mathbb{D}^2}$  where all  $h_j$  are holomorphic, it follows from Proposition 3.3 that  $\dim(Y_j \cap \mathbb{D}^2) = j$  for  $j = 0, 1, 2$ , and by Corollary 3.2 it follows that the graph of  $h$  over  $Y_j \setminus Y_{j-1}$  is totally real,  $j = 1, 2$ . Notice also that each  $Y_j$  is closed. It now suffices to show that

$$X_0 := \mathcal{G}_h(Y_{-3}) \subset \dots \subset X_5 := \mathcal{G}_h(Y_2)$$

fulfills the requirements of Theorem 3.5. We have seen that  $X_5$  is polynomially convex and that  $X_j \setminus X_{j-1}$  is a totally real manifold for  $j \geq 2$ . However,  $X_1 \setminus X_0$  is the graph of  $h$  over  $Y_{-2} \setminus Y_{-3} = (E \cup E') \setminus (B \cup B')$  which, by Lemma 4.1, is a  $\mathcal{C}^1$ -smooth manifold of real dimension 1. Hence,  $X_1 \setminus X_0$  is also totally real. It remains to see that  $\mathcal{C}(X_0) = \mathcal{O}(X_0)$ . Since  $X_0$  is the graph of  $h$  over  $Y_{-3}$ , it suffices to show that  $\mathcal{C}(Y_{-3}) = \mathcal{O}(Y_{-3})$ . Let  $\varphi \in \mathcal{C}(Y_{-3})$ , let  $\epsilon > 0$ , and let  $\tilde{f} \in \mathcal{O}(\Gamma^2)$  be such that  $|\varphi - \tilde{f}| < \epsilon/2$  on  $\Gamma^2$ . From Lemma 4.1 it follows that we, for any  $\delta > 0$ , can cover  $Y_{-3}$  by the



union of disjoint open sets  $U_0, \dots, U_\ell$  such that  $\text{diam}(U_j) < \delta$ ,  $j = 1, \dots, \ell$  and  $\sup_{x \in U_0} \text{dist}(x, \Gamma^2) < \delta$ . If  $\delta$  is sufficiently small it thus follows that  $\tilde{f} \in \mathcal{O}(U_0)$  and  $|\varphi - \tilde{f}| < \epsilon$  on  $U_0 \cap Y_{-3}$ . Moreover, perhaps after shrinking  $\delta$ , there are constant functions  $c_j$  that satisfies  $|c_j - \varphi| < \epsilon$  in  $U_j \cap Y_{-3}$ ,  $j = 1, \dots, \ell$ . We define  $f$  to be equal to  $\tilde{f}$  in  $U_0$  and  $c_j$  on  $U_j$ ,  $j = 1, \dots, \ell$ . Then  $f \in \mathcal{O}(Y_{-3})$  and  $|f - \varphi| < \epsilon$ .

It remains to prove Lemma 4.1. For fixed  $s_0 \in S^1$  there is a  $j$  such that  $g_j(\cdot, s_0)$  does not vanish identically since there is no holomorphic disk in  $\overline{\mathbb{D}}^2$  where all  $h_j$  are holomorphic. Hence, there is a neighborhood  $I_0$  of  $s_0$  in  $S^1$  such that  $g_j(\cdot, s)$  does not vanish identically for  $s \in I_0$ . We can thus find connected pairwise disjoint open  $I_1, \dots, I_\ell \subset S^1$  such that  $S^1 = \cup_j \bar{I}_j$  and for each  $i = 1, \dots, \ell$  there is a  $g_{j_i}$  such that  $g_{j_i}(\cdot, s)$  does not vanish identically for fixed  $s \in \bar{I}_i$ . We let

$$E_i = \{(z, s) \in \mathbb{D} \times \bar{I}_i; g_{j_i}(z, s) = 0\}, \quad E = \cup_{i=1}^\ell E_i;$$

clearly  $Z \subset E$ . We then let  $B_i$  be the union of  $E_i \cap (\mathbb{D} \times bI_i)$  and the set of points  $(z, s) \in E_i \cap (\mathbb{D} \times I_i)$  such that for every neighborhood  $V \ni (z, s)$ ,  $E_i \cap V$  is not a  $\mathcal{C}^1$ -smooth manifold of real dimension 1. Letting  $B = \cup_{i=1}^\ell B_i$  it follows that  $B \subset E$  is closed and that  $E \setminus B$  is a  $\mathcal{C}^1$ -smooth manifold of real dimension 1. Parts (i) and (ii) are proved.

Let  $A = \{(z, s) \in B; |z| \leq 1 - \delta\}$ . To prove part (iii) it suffices to cover  $A$  by the union of open sets  $U'_1, \dots, U'_k$  such that  $\text{diam}(U'_j) < \delta/2$  and  $B \cap (\cup_j bU'_j) = \emptyset$ . In fact, then we can take  $U_1 = U'_1$ ,  $U_2 = U'_2 \setminus \bar{U}_1$ ,  $U_3 = U'_3 \setminus (\bar{U}_1 \cup \bar{U}_2)$ , and so on; finally we take  $U_0 = \{(z, s) \in \mathbb{D} \times S^1; |z| > 1 - \delta\} \setminus (\cup_{j=1}^k \bar{U}_j)$ .

Fix  $s_0 \in S^1$ . Then there is a  $j$  such that  $s_0 \in \bar{I}_j$  and if  $s_0 \in I_j$  then  $s_0$  does not belong to any other  $I_i$ . Assume first that  $s_0 \in I_1$ . Then  $g_{j_1}(\cdot, s_0)$  does not vanish identically and we let  $\{g_{j_1}(\cdot, s_0) = 0\} \cap \{|z| \leq 1 - \delta\} = \{a_1(s_0), \dots, a_m(s_0)\}$ . Let  $V_j \subset \mathbb{D}$ ,  $j = 1, \dots, m$ , be pairwise disjoint open neighborhoods of  $a_j(s_0)$  such that  $\text{diam}(V_j) < \delta/10$  and  $g_{j_1}(\cdot, s_0)|_{bV_j} \neq 0$ . By Lemma 4.2 below, there is a neighborhood  $I_0 \subset I_1$  of  $s_0$  such that  $\text{diam}(I_0) < \delta/10$  and such that  $a_j(s)$ ,  $j = 1, \dots, m$ , is  $\mathcal{C}^1$ -smooth in a neighborhood of  $bI_0$  and  $g_{j_1}(\cdot, s)|_{bV_j} \neq 0$  for  $s \in \bar{I}_0$ . Letting  $U'_j = V_j \times I_0$  we see that  $A \cap (\mathbb{D} \times I_0)$  is covered by the union of the  $U'_j$ , that  $\text{diam}(U'_j) < \delta/2$ , and that  $\cup_j bU'_j \cap B = \emptyset$ . If instead  $s_0 \in bI_1$  then there is a unique  $j \neq 1$  such that also  $s_0 \in bI_j$ ; say that  $s_0 \in bI_1 \cap bI_2$ . Then neither  $g_{j_1}(\cdot, s_0)$  nor  $g_{j_2}(\cdot, s_0)$  vanishes identically and we can use the product  $g_{j_1} \cdot g_{j_2}$  in the above construction to find a neighborhood  $I_0 \subset S^1$  of  $s_0$  and finitely many  $U'_j$  covering  $A \cap (\mathbb{D} \times I_0)$ . By compactness of  $S^1$  we find the desired covering of  $A$ .

**Lemma 4.2.** *Let  $g \in \mathcal{C}^1(\mathbb{D} \times (-1, 1))$ . Assume that  $g(\cdot, t)$  is holomorphic for each fixed  $t \in (-1, 1)$  and that 0 is an isolated zero of  $g(\cdot, 0)$ . Let  $V$  be a*

neighborhood of 0 not containing any other zero of  $g(\cdot, 0)$  and assume that  $g(\cdot, 0)|_{bV} \neq 0$ . Then there is an  $\epsilon > 0$  and a closed subset  $K \subset (-\epsilon, \epsilon)$  without interior such that  $\{g(\cdot, t) = 0\} \cap V = \{a_1(t), \dots, a_m(t)\}$  for  $t \in (-\epsilon, \epsilon)$  and all  $a_j(t)$  are  $\mathcal{C}^1$ -smooth in  $(-\epsilon, \epsilon) \setminus K$ .

*Proof.* This lemma should be well known so we only sketch a proof. Let  $g'(z, t)$  denote the derivative of  $g$  with respect to  $z$ ; by the Cauchy integral formula it follows that  $g'(z, t) \in \mathcal{C}^1(\mathbb{D} \times (-1, 1))$ . The mapping

$$(4.1) \quad t \mapsto \frac{1}{2\pi i} \int_{bV} \frac{g'(z, t)}{g(z, t)} dz$$

is continuous for  $t$  close to 0 and takes values in  $\mathbb{N}$ ; thus it is constant. If it is 1 it follows that  $g(\cdot, t)$  has a simple zero,  $a(t)$ , in  $V$  for small fixed  $t$ . Then, by the residue theorem, it follows that

$$(4.2) \quad t \mapsto \frac{1}{2\pi i} \int_{bV} z \frac{g'(z, t)}{g(z, t)} dz$$

is equal to  $a(t)$ . Differentiating under the integral sign we see that  $a(t)$  is  $\mathcal{C}^1$ -smooth for  $t$  close to 0.

If the mapping (4.1) equals 2, then  $g(\cdot, t)$  has two zeroes,  $a_1(t), a_2(t)$ , possibly coinciding, in  $V$  for small fixed  $t$ . The mapping (4.2) now equals  $a_1(t) + a_2(t)$  and it is still  $\mathcal{C}^1$ -smooth. We say that  $t_0$  is *branching* if  $a_1(t_0) = a_2(t_0)$  and if there for every  $\delta > 0$  exists a  $t$  such that  $|t - t_0| < \delta$  and  $a_1(t) \neq a_2(t)$ . Let  $K$  be the set of branching  $t$ 's. More formally, one can define  $K$  as the boundary of the zero set of the  $\mathcal{C}^1$ -mapping

$$(4.3) \quad t \mapsto \frac{2}{2\pi i} \int_{bV} z^2 \frac{g'(z, t)}{g(z, t)} dz - \left( \frac{1}{2\pi i} \int_{bV} z \frac{g'(z, t)}{g(z, t)} dz \right)^2 = (a_1(t) - a_2(t))^2.$$

Then it is clear that  $K$  is closed and without interior. Let  $t_0$  be a point outside  $K$ . Then either (4.3) is non-zero in a neighborhood of  $t_0$  or (4.3) is 0 in a neighborhood of  $t_0$ . In the first case both  $a_1(t)$  and  $a_2(t)$  are simple zeroes of  $g(\cdot, t)$  for fixed  $t$  close to  $t_0$  and it follows from the first part of the proof that both  $a_1(t)$  and  $a_2(t)$  are  $\mathcal{C}^1$ -smooth close to  $t_0$ . In the second case we have that  $a_1(t) = a_2(t)$  for  $t$  close to  $t_0$  and so the  $\mathcal{C}^1$ -mapping (4.2) is equal to  $2a_1(t) = 2a_2(t)$  close to  $t_0$ .

The case when (4.1) equals  $m > 2$  is treated similarly.  $\square$

## 5. PROOF OF THEOREM 1.2

By a result of Tornehave (see *e.g.* Corollary 3.8.11 in Stout [11]) it is enough to prove that there exists such an *analytic* set  $Z$ .

Let  $h_j$  denote the pluriharmonic extension of  $f_j$  to  $\mathbb{D}^2$ , and write  $h = (h_1, \dots, h_N) : \overline{\mathbb{D}^2} \rightarrow \mathbb{C}^N$ . We let  $\mathcal{G}_h(\overline{\mathbb{D}^2})$  denote the graph of  $h$  over  $\overline{\mathbb{D}^2}$  in  $\mathbb{C}^2 \times \mathbb{C}^N$ , and we let  $\mathcal{G}_h(\Gamma^2)$  denote the graph over  $\Gamma^2$ . Since  $\Gamma^2$  is totally real it suffices to show that *either  $\mathcal{G}_h(\Gamma^2)$  is polynomially convex, or there*

exists a variety  $Z \subset \overline{\mathbb{D}^2} \setminus \Gamma^2$  with  $\overline{Z} \setminus Z \subset \Gamma^2$ , all  $h_j$ -s holomorphic on  $Z$ , and  $\mathcal{G}_h(Z) \subset \widehat{\mathcal{G}_h(\Gamma^2)}$ . We assume that  $\mathcal{G}_h(\Gamma^2)$  is not polynomially convex, and proceed to find a variety  $Z$ . We will consider different possibilities through some lemmas, and then we will sum up the entire argument in the end.

The first thing we want to show is that

**Lemma 5.1.** *Either  $\widehat{\mathcal{G}_h(\Gamma^2)}$  contains a holomorphic disk  $\mathcal{G}_h(\Delta)$ , where  $\Delta$  is one of the disks in  $b\mathbb{D}^2 \setminus \Gamma^2$ , or  $\widehat{\mathcal{G}_h(\Gamma^2)} \cap (\mathcal{G}_h(b\mathbb{D}^2 \setminus \Gamma^2)) = \emptyset$ .*

*Proof.* Choose a point  $\zeta = (\zeta_1, \zeta_2) \in b\mathbb{D}^2 \setminus \Gamma^2$  with  $\mathcal{G}_h(\zeta) \in \widehat{\mathcal{G}_h(\Gamma^2)}$ . Without loss of generality we assume that  $|\zeta_1| = 1$ . Let  $\Delta$  be the disk  $\Delta := \{(\zeta_1, w) : |w| < 1\}$ . Since each point of  $b\mathbb{D}_{z_1}$  is a peak point for  $\mathcal{O}(\mathbb{C}_{z_1})|_{\overline{\mathbb{D}}}$  it follows that  $\mathcal{G}_h(\zeta) \in \widehat{\mathcal{G}_h(b\Delta)}$ . By Wermer [15] we have that  $h_j$  is holomorphic on  $\Delta$  for  $j = 1, \dots, N$ .  $\square$

As we proceed we assume that there is no such disk  $\Delta$ , i.e., if we can locate a closed variety in  $\widehat{\mathcal{G}_h(\Gamma^2)} \cap \mathcal{G}_h(\mathbb{D}^2)$ , then it is automatically attached to  $\mathcal{G}_h(\Gamma^2)$ .

Let

$$\tilde{Z} := \{z \in \mathbb{D}^2 : \bar{\partial}h_{i_1} \wedge \bar{\partial}h_{i_2}(z) = 0, \forall (i_1, i_2), 1 \leq i_1, i_2 \leq N\}.$$

By Lemma 3.6 we have that  $\mathcal{G}_h(\overline{\mathbb{D}^2})$  is polynomially convex, and since  $\mathcal{G}_h(\mathbb{D}^2 \setminus \tilde{Z})$  is totally real by Lemma 3.1, it follows from Theorem 3.5 that

$$\widehat{\mathcal{G}_h(\Gamma^2)} \subset \mathcal{G}_h(\Gamma^2 \cup \tilde{Z}).$$

(Because all totally real points are peak-points.)

**Lemma 5.2.** *Assume that  $\tilde{Z} \neq \mathbb{D}^2$ , let  $Z_\alpha$  be an irreducible component of  $\tilde{Z}$  of dimension 1, and let  $z_0 \in Z_\alpha \cap \tilde{Z}_{reg}$  with  $\mathcal{G}_h(z_0) \in \widehat{\mathcal{G}_h(\Gamma^2)}$ . Then  $\mathcal{G}_h(Z_\alpha) \subset \widehat{\mathcal{G}_h(\Gamma^2)}$ . All the  $h_j$ -s are holomorphic along  $Z_\alpha$ .*

*Proof.* Let  $\Omega$  be a small open neighborhood of  $z_0$  such that  $\Omega \cap Z_\alpha$  is a smooth disk  $D_{z_0}$ . Then, if  $\Omega$  is small enough, we have that  $K := (b\Omega \times \mathbb{C}^N) \cap \widehat{\mathcal{G}_h(\Gamma^2)} \subset \mathcal{G}_h(bD_{z_0})$ , and so by Rossi's local maximum principle

$$\mathcal{G}_h(z_0) \in \hat{K} \subset \widehat{\mathcal{G}_h(bD_{z_0})}.$$

By Wermer's maximality theorem it follows that all  $h_j$ -s are holomorphic on  $Z_\alpha$  near  $z_0$ , and then  $\mathcal{G}_h(D_{z_0})$  is contained in the hull, since we must have that  $K = \mathcal{G}_h(bD_{z_0})$ . Since this holds near any point of  $Z_\alpha \cap \tilde{Z}_{reg}$  in the hull, it follows that  $Z_\alpha \cap \tilde{Z}_{reg}$  is contained in the hull, hence  $Z_\alpha$  is contained in the hull.  $\square$

Finally we need to consider the case that  $\tilde{Z} = \mathbb{D}^2$ . We want to change coordinates. For each  $j$  let  $g_j \in \mathcal{O}(\mathbb{D}^2)$  with  $\text{Im}(g_j) = \text{Im}(h_j)$ , and let  $\varphi_j \in \mathcal{O}(\mathbb{D}^2)$  with  $\text{Re}(\varphi_j) = u_j := h_j - g_j$ . Then  $u_j$  is real for all  $j$ , and for any compact set  $K \subset \mathbb{D}^2$  we have that  $\mathcal{G}_h(K)$  being polynomially convex is equivalent to  $\mathcal{G}_u(K)$  being polynomially convex. We will show that  $\widehat{\mathcal{G}_h(\Gamma^2)}$  contains the graph of a leaf of the (possibly singular) Levi-foliation of a level set  $\{u_j = c\}$  for at least one  $j$ . We want to use our coordinate change to study the hull, but since the  $u_j$ -s are not necessarily continuous up to  $\Gamma^2$  we will consider a certain exhaustion of  $\mathbb{D}^2$ .

By the assumption that  $\widehat{\mathcal{G}_h(\Gamma^2)} \cap (\mathcal{G}_h(b\mathbb{D}^2 \setminus \Gamma^2)) = \emptyset$  there exist sequences of real numbers  $r_j \rightarrow 1, \epsilon_j \rightarrow 0$ , such that the following holds: defining

$$Q_j := \{(z_1, z_2) : |z_1| = r_j, r_j - \epsilon_j \leq |z_2| \leq r_j\} \\ \cup \{(z_1, z_2) : |z_2| = r_j, r_j - \epsilon_j \leq |z_1| \leq r_j\}$$

we have that

$$1) \quad \tilde{K}_j := \widehat{\mathcal{G}_h(\Gamma^2)} \cap \mathcal{G}_h(b\mathbb{D}_{r_j}^2) \subset \mathcal{G}_h(Q_j).$$

Let  $K_j$  denote the projection of  $\tilde{K}_j$  to  $\mathbb{C}^2$ . Then the  $h_j$ -s, and consequently the  $u_i$ -s, are smooth in a neighborhood of  $K_j$  for all  $j$ .

**Lemma 5.3.** *Assume that there is a subsequence  $r_{j_k}$  of  $r_j$  such that for each  $k$  there exists a point  $\mathcal{G}_h((a_k, b_k)) \in \widehat{\mathcal{G}_h(Q_{j_k})}$  with  $|a_k| = r_{j_k}$  and  $|b_k| < r_{j_k} - \epsilon_{j_k}$ . Then there exists a disk  $\mathbb{D}_a = \{a\} \times \mathbb{D}$  with  $|a| = 1$ , and all  $h_j$ -s holomorphic on  $\mathbb{D}_a$ .*

*Proof.* This is similar to the proof of Lemma 5.1. Since each point  $a_k$  is a peak point for  $\mathcal{O}(\overline{\mathbb{D}_{|a_k|}})$  it follows from [1] (alternatively Wermer's Maximality Theorem combined with Theorem 1.3) that all  $h_j$ -s are holomorphic on  $\mathbb{D}_{a_k}$ . By passing to a subsequence we may assume that  $a_k \rightarrow a \in b\mathbb{D}$  as  $k \rightarrow \infty$ .  $\square$

We will now make the assumption that

- 2) none of the functions  $h_j$  are holomorphic on any of the disks in  $b\mathbb{D}_{r_j}^2 \setminus Q_j$ , or, equivalently, for any point  $x \in b\mathbb{D}_{r_j}^2 \setminus Q_j$  we have that  $\mathcal{G}_h(x) \notin \widehat{\mathcal{G}_h(Q_j)}$ .

We will now consider the case that  $\tilde{Z} = \mathbb{D}^2$  and that at least one of the  $u_j$ -s, say  $u_1$ , is non-constant. Define  $L_c = L_c(u_1) := \{z \in \mathbb{D}^2 : u_1(z) = c\}$ .

We have that

$$L_c(u_1) = \bigcup_{r \in \mathbb{R}} L_{c+i \cdot r}(\varphi_1),$$

so  $L_c$  is the disjoint union of analytic sets, which we will call leaves of the (singular) lamination  $L_c$ .

**Lemma 5.4.** *There exists a discrete set  $A \subset \mathbb{D}^2$  such that near any point  $z_0 \in \mathbb{D}^2 \setminus A$ , the set  $\{\varphi_1(z) = \varphi_1(z_0)\}$  is a smooth surface.*

*Proof.* Let

$$W = \{\partial\varphi_1 = 0\}.$$

Assume for simplicity that  $W$  is a connected 1-dimensional variety with  $\varphi_1|_W \equiv 0$ . Let  $\tilde{W} := \{\varphi_1 = 0\}$ . Then  $A = \tilde{W}_{\text{sing}} \cap W$  is a discrete set.  $\square$

**Lemma 5.5.** *Assume that  $\tilde{Z} = \mathbb{D}^2$ . Assume that  $\mathcal{G}_h(x_0) \in \widehat{\mathcal{G}_h(Q_j)} \cap \mathcal{G}_h(\mathbb{D}_{r_j})$  and set  $c_1 = u_1(x_0)$ . Then there exists a point  $z_0 \in (L_{c_1} \setminus A) \cap \mathbb{D}_{r_j}^2$  with  $\mathcal{G}_h(z_0) \in \widehat{\mathcal{G}_h(Q_j)}$ . Moreover, for any point  $z_0 \in L_{c_1} \setminus A$ , we have that  $\mathcal{G}_h(z_0) \in \widehat{\mathcal{G}_h(Q_j)}$  implies that  $\mathcal{G}_h(\mathcal{L}_{z_0} \cap \mathbb{D}_{r_j}^2) \subset \widehat{\mathcal{G}_h(Q_j)}$  with  $h$  holomorphic along  $\mathcal{L}_{z_0}$ , where  $\mathcal{L}_{z_0}$  denotes the leaf through  $z_0$ .*

*Proof.* We have that  $\mathcal{G}_h(z_0) \in \widehat{\mathcal{G}_h(Q_j)}$  is equivalent to  $\mathcal{G}_u(z_0) \in \widehat{\mathcal{G}_u(Q_j)}$ .

Let  $c = u(z_0)$  and  $Q_j^c := \{z \in Q_j : h(z) = c\}$ . By Proposition 3.7 we have that

$$\mathcal{G}_u(z_0) \in \widehat{\mathcal{G}_u(Q_j^c)} \subset L_c.$$

Since  $Q_j^c$  is a level set of  $u$  this means that  $z_0 \in \widehat{Q_j^c}$ . It also follows that the hull contains graphs over the regular points, because the singular set is discrete.

**Lemma 5.6.** *Let  $z_0 \in L_{c_1} \cap (\mathbb{D}_{r_j}^2 \setminus A)$  and let  $\mathcal{L}_{z_0}$  denote the leaf through  $z_0$ . Then  $z_0 \in \widehat{Q_j^c}$  if and only if  $\mathcal{L}_{z_0} \subset \widehat{Q_j^c}$ .*

*Proof.* This is quite similar to the proof of Lemma 5.2. Choose a small neighborhood  $\Omega$  of  $z_0$  such that  $D_{z_0} = \Omega \cap \mathcal{L}_{z_0}$  is a smoothly bounded disk. We may assume that  $\varphi_1(z_0) = 0$  and  $\varphi_1((L_c \cap \overline{\Omega}) \setminus D_{z_0}) \subset i \cdot \mathbb{R} \setminus \{0\}$ .

Let  $K := \widehat{Q_j^c} \cap b\Omega$ . Then  $z_0 \in \widehat{K}$  by Rossi's principle. This implies that  $bD_{z_0} \subset K$ . Otherwise, let  $K_{z_0} := bD_{z_0} \cap K$  and let  $g$  be holomorphic with  $|g(z_0)| > \|g\|_{K_{z_0}}$ . Let  $\tau$  be holomorphic on  $\mathbb{C}$ ,  $\tau(0) = 1$ ,  $0 < \tau(z) < 1$  for  $z \in i \cdot \mathbb{R} \setminus \{0\}$ . Then  $g \cdot (\tau \circ \varphi_1)^m$  separates  $z_0$  from  $K$  for large  $m$ . This is a contradiction.

Then clearly  $D_{z_0}$  is contained in the hull, and since  $K_{z_0} = bD_{z_0}$  we have that  $u_2, \dots, u_N$  are also constant on  $D_{z_0}$ .  $\square$

Finally note that if  $\mathcal{L}_{z_0} \subset \widehat{Q_j^c}$ , then  $\mathcal{G}_u(\mathcal{L}_{z_0}) \subset \widehat{\mathcal{G}_u(Q_j^c)}$ .  $\square$

*Proof of Theorem 1.2:*

Let  $h_j$  be the pluriharmonic extension of  $f_j$  to  $\mathbb{D}^2$  for  $j = 1, \dots, N$ . Clearly we may assume that not all  $h_j$ -s are constant. If all  $h_j$ -s are holomorphic on a disk  $\triangle$  in  $b\mathbb{D}^2 \setminus \Gamma^2$  the algebra is clearly not generated, and the conclusion

of the theorem would hold. So from now on we assume that not all of the functions are holomorphic any of these disks. By Lemma 5.1 none of these disks intersect the hull.

Consider

$$\tilde{Z} := \{z \in \mathbb{D}^n : \bar{\partial}h_{i_1} \wedge \bar{\partial}h_{i_2}(z) = 0, \forall (i_1, i_2), 1 \leq i_1, i_2 \leq N\}.$$

Assume that  $\tilde{Z} \neq \mathbb{D}^2$ . If  $\mathcal{G}_h(\Gamma^2)$  is not polynomially convex, then according to Lemma 5.2 there is an irreducible component  $Z_\alpha$  of  $\tilde{Z}$  on which all  $h_j$ -s are holomorphic in the hull. By assumption  $\overline{Z}_\alpha \cap b\mathbb{D}^2 \subset \Gamma^2$ .

Finally we consider the case that  $\tilde{Z} = \mathbb{D}^2$ . We assume that  $h_1$  is non-holomorphic. Let  $\mathcal{G}_h(x_0) \in \widehat{\mathcal{G}_h(\Gamma^2)}$ . For  $j$  large enough such that  $x_0 \in \mathbb{D}_{r_j}^2$  we then have that  $\mathcal{G}_h(x_0) \in \widehat{\mathcal{G}_h(Q_j)}$  (by 1) and Rossi's principle).

By Lemma 5.5 we may assume that  $x_0 \notin A$ . Assume that  $\varphi_1(x_0) = 0$  and let  $Z$  denote the irreducible component of  $\{z \in \mathbb{D}^2 : \varphi_1(z) = 0\}$  containing  $x_0$ . According to Lemma 5.5 we have that  $\mathcal{G}_h(Z \cap \mathbb{D}_{r_j}^2) \subset \widehat{\mathcal{G}_h(Q_j)}$  for all  $j$ . It follows that  $G_h(Z) \subset \widehat{\mathcal{G}_h(\Gamma^2)}$ . □

**Example 5.7.** Let  $f(z_1, z_2) := \operatorname{Re}(z_1 + c \cdot z_2)$  with  $c \in \mathbb{C}^*$ . Since  $f$  is real and pluriharmonic on  $\mathbb{C}^2$  we set  $h = f$ . We will show that  $[z_1, z_2, f]_{\Gamma^2} = \mathcal{C}(\Gamma^2)$ . Otherwise, by Theorem 1.2, there is a non-trivial closed variety  $Z \subset \mathbb{D}^2$  with  $\overline{Z} \setminus Z \subset \Gamma^2$  on which  $h$  is holomorphic. Then  $h$  is constant  $Z$ , so  $Z$  is contained in a hypersurface

$$\{z_1 + c \cdot z_2 = r + i \cdot s : r \text{ fixed and } s \in \mathbb{R}\}.$$

But then  $Z$  is contained in a complex line

$$L_k := \{z_1 + c \cdot z_2 = k : k \in \mathbb{C}\}.$$

Since  $L_k \cap b\mathbb{D}^2$  is not contained in  $\Gamma^2$  this is impossible.

## 6. PROOF OF THEOREM 1.3

Assume that there is no analytic disk in  $\Omega$  where all the  $h_j$  are holomorphic. We will prove that the polynomials in  $\mathbb{C}^n \times \mathbb{C}^N$  are dense in the uniform algebra of continuous functions on the graph  $\mathcal{G}_h(\overline{\Omega})$ . By Lemma 3.6 we have that  $\mathcal{G}_h(\overline{\Omega})$  is polynomially convex.

We will define a stratification of  $\mathcal{G}_h(\overline{\Omega})$  so that we can use Theorem 3.5. As in Section 4 we begin by stratifying  $\overline{\Omega}$ : Let

$$\begin{aligned} Y_n &= \overline{\Omega}, \\ Y_{n-1} &= b\Omega \cup \Omega_H^n =: b\Omega \cup Z_{n-1}; \end{aligned}$$

see Subsection 3.2 for the definition of  $\Omega_H^n$ . Since there is no analytic disk in  $\Omega$  where all  $h_j$  are holomorphic, it follows from Proposition 3.3 that

$\dim Z_{n-1} \leq n - 1$ . For an analytic set  $V$  we write  $V'$  for the union of the irreducible components of  $V$  of maximal dimension and  $V''$  for the union of the rest of the components. Let

$$\begin{aligned} Y_{n-2} &= b\Omega \cup \text{Sing}(Z_{n-1}) \cup Z''_{n-1} \cup \text{Reg}(Z'_{n-1})_H^{n-1} \\ &=: b\Omega \cup Z_{n-2}. \end{aligned}$$

Again, from Proposition 3.3 it follows that  $\dim Z_{n-2} \leq n - 2$ ; notice that  $\dim (Z_{n-1})_{\text{sing}} \leq n - 2$  and  $\dim Z''_{n-1} \leq n - 2$  automatically. We define recursively

$$\begin{aligned} Y_k &= b\Omega \cup \text{Sing}(Z_{k+1}) \cup Z''_{k+1} \cup \text{Reg}(Z'_{k+1})_H^{k+1} \\ &=: b\Omega \cup Z_k \end{aligned}$$

and from Proposition 3.3 we have  $\dim Z_k \leq k$ . Moreover,  $Y_k \setminus Y_{k-1}$  is (either empty or) a  $k$ -dimensional complex manifold where  $H_k$  is non-vanishing, and so, by Corollary 3.2, the graph of  $h|_{Y_k \setminus Y_{k-1}}$  is a totally real manifold. We define our stratification of  $\mathcal{G}_h(\overline{\Omega})$  as follows:

$$X_0 := \mathcal{G}_h(Y_0) \subset \cdots \subset X_n := \mathcal{G}_h(Y_n).$$

We have seen that  $X_n$  is polynomially convex, that  $X_k \setminus X_{k-1}$  is a totally real manifold, and that  $X_0$  is the union of the image of  $b\Omega$  and a discrete set in  $\Omega$ . Since by assumption  $[z_1, \dots, z_n, h_1, \dots, h_N]_{b\Omega} = \mathcal{C}(b\Omega)$  it follows that  $\mathcal{O}(X_0)$  is dense in  $\mathcal{C}(X_0)$ . By Theorem 3.5 we are thus done.

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